Lecture 10: Least Squares Bases

Today:

- See the value of ***orthogonal bases*** for least squares function approximation

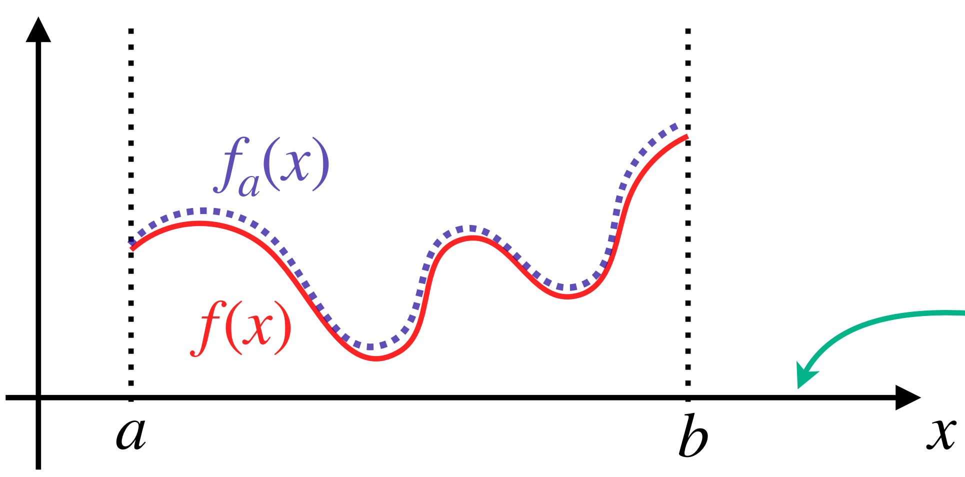
Reminder from Wednesday

In ***least squares approximation,*** we wrote our approximation function as a linear combo of basis functions

$$f_a(x) = \sum_{j=0}^{n} c_j b_j(x) \qquad (1)$$

And arrived at a linear system for computing these coefficients

$$\begin{bmatrix} (b_0, b_0) & (b_1, b_0) & \cdots & (b_{n-1}, b_0) & (b_n, b_0) \\ (b_0, b_1) & (b_1, b_1) & \cdots & (b_{n-1}, b_1) & (b_n, b_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (b_0, b_{n-1}) & (b_1, b_{n-1}) & \cdots & (b_{n-1}, b_{n-1}) & (b_n, b_{n-1}) \\ (b_0, b_n) & (b_1, b_n) & \cdots & (b_{n-1}, b_n) & (b_n, b_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} (f, b_0) \\ (f, b_1) \\ \vdots \\ (f, b_{n-1}) \\ (f, b_n) \end{bmatrix} \qquad (2)$$



This system came from imposing the least squares condition $\min_{f_a \in \mathcal{V}} ||f - f_a||^2$ and using the fact that the approximation error had to be orthogonal to the subspace!

Intuition: don't explicitly require $f = f_a$ at any specific points. Just try to get an f_a that is close to f in a meaningful way.

Today. What basis functions should we use to solve for (2) as efficiently as possible?

Orthogonal bases are best

Remember (week 1) that by definition, an ***orthogonal basis*** $\mathscr{B} = \{b_0, \ldots, b_n\}$ for a space $\mathscr{V}$ satisfies

$$(b_i, b_j) = \begin{cases} \alpha_i & i = j \\ 0 & \text{else} \end{cases}$$

For some scalar numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$.

That means that the matrix system $\mathbf{Ac} = \mathbf{b}$ from equation (1) simplifies to

$$\mathbf{Dc} = \mathbf{b}$$

Where $\mathbf{D}$ is a diagonal matrix with zero entries off the diagonal and the j^{th} diagonal entry equal to α_j

$\implies \quad \alpha_j c_j = (f, b_j)$

$\implies \quad$ solving the linear system (1) is trivial!

Punchline. We want orthogonal bases when doing least squares approximation

We will consider a specific example of an orthogonal basis today

But what if we don't know an orthogonal basis off the cuff??

- There is a general way of constructing an orthogonal basis from a non-orthogonal basis (HW)

An example of an orthogonal basis

Consider the subspace $\mathscr{T}^n[-L, L]$ for some real number L.

We showed last week that a basis for the subspace is

$$\mathscr{B} = \left\{ 1, \exp\left(\frac{2}{L} i\pi x\right), \exp\left(-\frac{2}{L} i\pi x\right), \exp\left(\frac{4}{L} i\pi x\right), \exp\left(-\frac{4}{L} i\pi x\right), \ldots, \exp\left(\frac{2n}{L} i\pi x\right), \exp\left(-\frac{2n}{L} i\pi x\right) \right\}$$

Magic. Defining the inner product

$$(f, g) = \int_{-L}^{L} f(x)\overline{g(x)} dx$$

Remember the overbar means complex conjugate!
(This is necessary to satisfy the rules of an IP because the exp functions are complex!)

we have for some integers j, k between $-n$ and n

$$\left(\exp\left(\frac{2j}{L} i\pi x\right), \exp\left(\frac{2k}{L} i\pi x\right) \right) = \int_{-L}^{L} \exp\left(\frac{2j}{L} i\pi x\right) \exp\left(-\frac{2k}{L} i\pi x\right) dx$$

$$= \int_{-L}^{L} \exp\left(\frac{2j - 2k}{L} i\pi x\right) dx$$

$$= \begin{cases} 2L & j = k \\ 0 & \text{else} \end{cases} \qquad (3)$$

Activity

Using equation (3), develop an expression for the unknown coefficients c_j in terms of $f(x)$ and the basis functions contained within $\mathscr{B}$. Use this expression for the coefficients to obtain an expression for $f_a(x)$

Answer.

Since the basis is orthogonal, the linear system (1) simplifies considerably and we can solve for the coefficients directly:

$$c_j = \frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-\frac{2j}{L}i\pi x\right)dx$$

And we therefore have an expression for our approximation to $f_a(x)$:

$$f_a(x) = \sum_{j=-n}^{n}\underbrace{\left[\frac{1}{2L}\int_{-L}^{L} f(x)\exp\left(-\frac{2j}{L}i\pi x\right)dx\right]}_{c_j}\exp\left(\frac{2j}{L}i\pi x\right)$$

Questions?

We found that the Lagrange basis functions lead to an identity matrix when solving for the coefficients in interpolation. Does this mean these basis functions are orthogonal?

No! The matrix for interpolation was formed from evaluating the basis function at specific points, NOT by taking inner products between basis functions

Which is more accurate: least squares approximation or interpolation?

For a given choice of subspace, least squares approximation is optimal. This is by construction! Least squares satisfies $\min_{f_a \in \mathcal{V}} || f - f_a ||$, so there can not be another approximating function of that subspace that does better!

So why do interpolation?

Many factors go into picking a method! One example: maybe it's hard to evaluate the inner products or to construct an orthogonal basis. So maybe it's actually faster to use interpolation for a higher value of n than for least squares. These issues are problem dependent. That's why we're learning ***all*** of these methods!