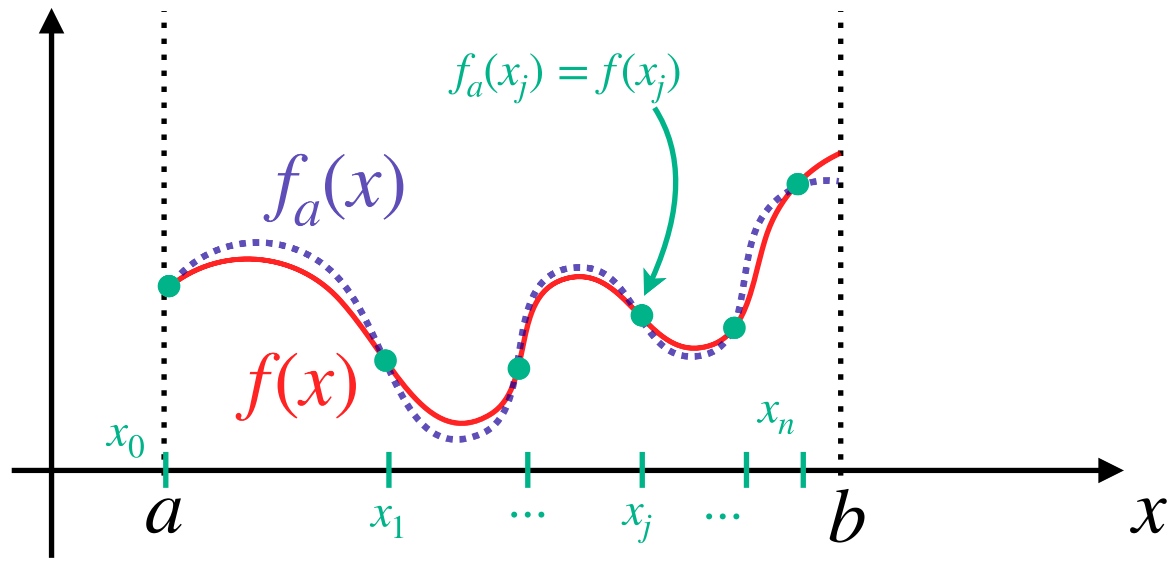
Lecture 7: Trigonometric Interpolation

Today:

- Approximating with ***trigonometric functions*** rather than polynomials

Where are we?

Recap: Thus far, we have dealt with approximating a function $f(x)$ using interpolation with polynomials, either globally (monomial & Lagrange bases), or locally (cubic splines)



This time:

(A) We will continue to use interpolation to approximate $f(x)$

(B) Now, we will consider interpolation onto a subspace defined by *trigonometric functions* (i.e., sines and cosines)

(C) This is a useful approach when you known your function is periodic

(D) We will only consider global interpolation in this trigonometric setting

A suitable basis for trigonometric interpolation

Our approach will be similar to that of polynomial interpolation.

We first pick a subspace: $\mathscr{T}^n[a, b]$

The vector space defined by functions that have between 1 to n integer periods on $[a, b]$

We then need a ***basis*** for this subspace:

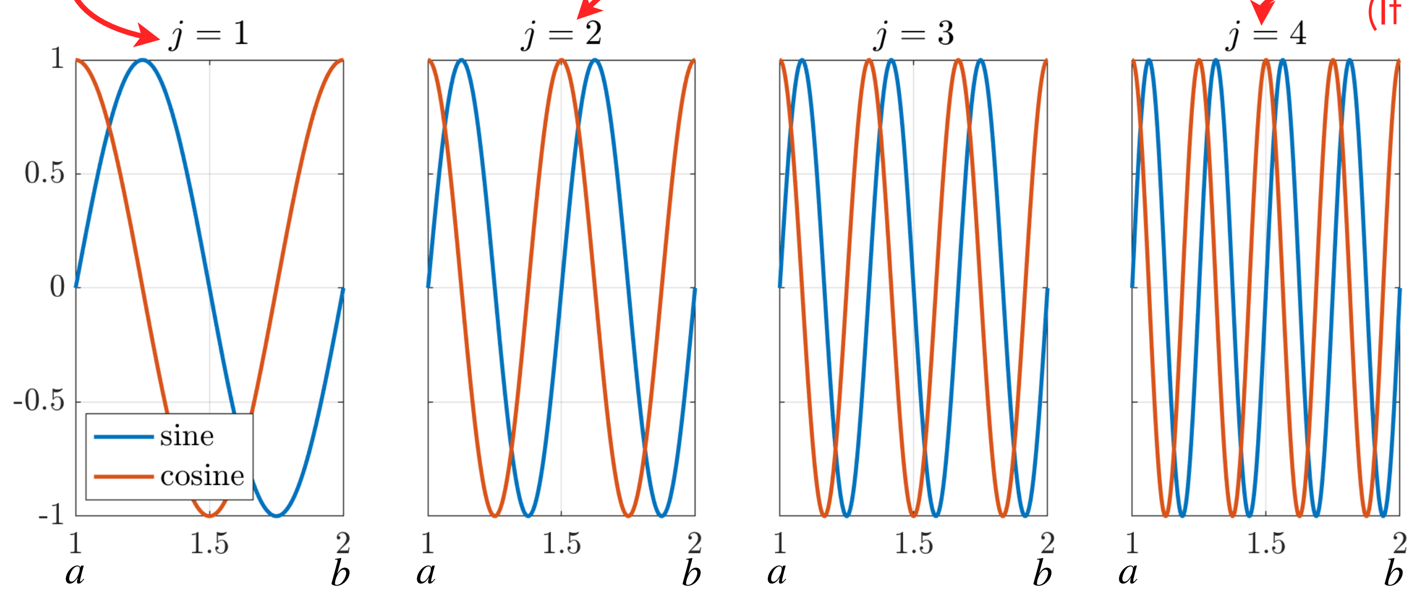
It turns out that one basis is

$$\mathscr{B}_1 = \left\{ 1, \cos\left(\frac{2\pi(x-a)}{(b-a)}\right), \sin\left(\frac{2\pi(x-a)}{(b-a)}\right), \cos\left(\frac{4\pi(x-a)}{(b-a)}\right), \sin\left(\frac{4\pi(x-a)}{(b-a)}\right), \ldots, \cos\left(\frac{2\pi n(x-a)}{(b-a)}\right), \sin\left(\frac{2\pi n(x-a)}{(b-a)}\right) \right\}$$

(If $n = 4$)

Can write as

$$\cos\left(\frac{2\pi j(x-a)}{b-a}\right), \quad \sin\left(\frac{2\pi j(x-a)}{b-a}\right)$$



Intuition for why $\mathscr{B}_1$ is a basis:

- Clearly the cosine/sine pair for each j has j integer periods over the interval
- The sines let us represent functions with zero values at the ends, and the cosines for functions with nonzero end values

We will use a different but strongly related basis

Another basis for $\mathscr{T}^n[a,b]$ turns out to be

$$\mathscr{B}_2 = \left\{1, \exp\left(\frac{2\pi i(x-a)}{(b-a)}\right), \exp\left(-\frac{2\pi i(x-a)}{(b-a)}\right), \exp\left(\frac{4\pi i(x-a)}{(b-a)}\right), \exp\left(-\frac{4\pi i(x-a)}{(b-a)}\right), \ldots, \exp\left(\frac{2\pi i n(x-a)}{(b-a)}\right), \exp\left(-\frac{2\pi i n(x-a)}{(b-a)}\right)\right\}$$

Some important questions.

- Note: $\exp(ikx) = \cos(kx) + i\sin(kx) \implies$ the basis functions are complex-valued!

  How do we handle that?

- Can we relate $\mathscr{B}_1$ to $\mathscr{B}_2$?

- Why would we use $\mathscr{B}_2$?

Worked example: working with $\mathscr{B}_2$

Worked example. Let's say that when representing a function $f(x)$ with $\mathscr{B}_2$, the terms associated with $j = 1$ (the functions with one integer period) are

$$c_1 \exp\left(\frac{2\pi i(x-a)}{(b-a)}\right) + c_{-1} \exp\left(-\frac{2\pi i(x-a)}{(b-a)}\right)$$

What can we say about the coefficients c_1, c_{-1}?

$$c_1\left[\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\sin\left(\frac{2\pi(x-a)}{(b-a)}\right)\right] + c_{-1}\left[\cos\left(-\frac{2\pi(x-a)}{(b-a)}\right) + i\sin\left(-\frac{2\pi(x-a)}{(b-a)}\right)\right]$$

$$\implies c_1\left[\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\sin\left(\frac{2\pi(x-a)}{(b-a)}\right)\right] + c_{-1}\left[\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) - i\sin\left(\frac{2\pi(x-a)}{(b-a)}\right)\right]$$

$$\implies (c_1 + c_{-1})\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\,(c_1 - c_{-1})\sin\left(\frac{2\pi(x-a)}{(b-a)}\right) \qquad (1)$$

Assuming $f(x)$ is real, then when (1) gets evaluated the result must be real valued!

$\implies$ c_1, c_{-1} must be complex. (i.e., $c_1 = c_1^R + ic_1^I$ and $c_{-1} = c_{-1}^R + ic_{-1}^I$)

Activity: Show that for (1) to be real, c_1, c_{-1} must satisfy $c_1 = \overline{c_{-1}}$

c_1 must be the complex conjugate of c_{-1}

Worked example: working with $\mathscr{B}_2$ (cont).

Activity: Show that for (1) to be real, c_1, c_{-1} must satisfy $c_1 = \overline{c_{-1}}$

$$\implies (c_1^R + ic_1^I + c_{-1}^R + ic_{-1}^I)\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\,(c_1^R + ic_1^I - c_{-1}^R - ic_{-1}^I)\sin\left(\frac{2\pi(x-a)}{(b-a)}\right)$$

$$\implies (c_1^R + c_{-1}^R)\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\,(c_1^I + c_{-1}^I)\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + i\,(c_1^R - c_{-1}^R)\sin\left(\frac{2\pi(x-a)}{(b-a)}\right) - (c_1^I - c_{-1}^I)\sin\left(\frac{2\pi(x-a)}{(b-a)}\right)$$

$c_1^I = -c_{-1}^I$ $\qquad$ $c_1^R = c_{-1}^R$

$$\implies \quad c_1 = \overline{c_{-1}}$$

When c_1, c_{-1} satisfy this property, (1) simplifies to the real-valued function

$$2c_1^R\cos\left(\frac{2\pi(x-a)}{(b-a)}\right) - 2c_1^I\sin\left(\frac{2\pi(x-a)}{(b-a)}\right) \qquad (2)$$

Worked example: relating $\mathscr{B}_1$ to $\mathscr{B}_2$

Worked example. Let's say that when representing a function $f(x)$ with $\mathscr{B}_1$, the terms associated with $j = 1$ (the functions with one integer period) are

$$a_1 \cos\left(\frac{2\pi(x-a)}{(b-a)}\right) + b_1 \sin\left(\frac{2\pi(x-a)}{(b-a)}\right)$$

Using your answer from the last worked example, relate c_1 to a_1, b_1.

Remember equation (2) from the last worked example:

$$\implies 2c_1^R \cos\left(\frac{2\pi(x-a)}{(b-a)}\right) -2c_1^I \sin\left(\frac{2\pi(x-a)}{(b-a)}\right) \quad (2)$$

$$\implies a_1 = 2c_1^R,\ b_1 = -2c_1^I$$

So if we believe that we can use $\mathscr{B}_1$ as a basis, we can use $\mathscr{B}_2$

And we have some intuition for dealing with these scary-seeming coefficients

Summary: how to use $\mathscr{B}_2$ as a basis for $\mathscr{T}^n[a, b]$

So if we believe that we can use $\mathscr{B}_1$ as a basis, we can use $\mathscr{B}_2$

And we have some intuition for dealing with these scary-seeming coefficients

Key takeaway. When approximating a function $f(x)$ using $\mathscr{T}^n[a, b]$, we can write the approximation as a linear combination of the basis functions in $\mathscr{B}_2$:

$$f_a(x) = \sum_{k=-n}^{n} c_k \exp\left(\frac{2\pi i k(x-a)}{(b-a)}\right)$$

where $c_k = \overline{c_{-k}}$ for $j = 1,\ldots,n$ (provided $f(x)$ is a real-valued function).

Recap: return to the questions highlighted in slide 4

Some important questions.

- Note: $\exp(ikx) = \cos(kx) + i\sin(kx) \implies$ the basis functions are complex-valued!

  How do we handle that?

  Make sure $c_j = \overline{c_{-j}}$ ✓

- Can we relate $\mathscr{B}_1$ to $\mathscr{B}_2$?

  Yes! The coefficients associated with the index j basis functions are related by $a_j = 2c_j^R$, $b_j = -2c_j^I$ ✓

- Why would we use $\mathscr{B}_2$?

  We will discuss this next

  We will first talk about solving for the c_j using interpolation

  Then discuss the benefit of $\mathscr{B}_2$: lets us solve for the coefficients FAST with the "Fast Fourier Transform"

Solving for the coefficients for $\mathscr{B}_2$

We will use ***function interpolation***. Using our subspace, $\mathscr{T}^n[a,b]$, and our basis for that subspace $\mathscr{B}_2$...

Necessary because we have unknown coeffs $c_{-n}, \ldots, c_0, \ldots, c_n$
Label interpolation points as $x_0, \ldots, x_{2n}$

(A) Break domain $[a,b]$ up into $2n+1$ interpolation points

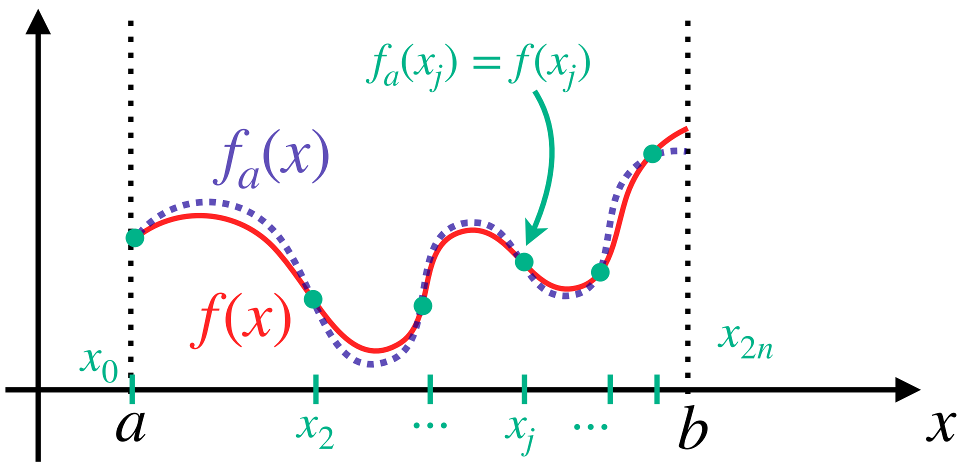
(B) Require: approximate function equals the true function at the interpolation points

To make notation easier, write as $k\xi_j$
i.e., $\xi_j = \dfrac{2\pi i(x_j - a)}{(b-a)}$

$$f_a(x_j) = f(x_j), \quad j = 0, \ldots, 2n$$

$$\implies \sum_{k=-n}^{n} c_k \exp\left(\frac{2\pi i k(x_j - a)}{(b-a)}\right) = f(x_j), \quad j = 0, \ldots, 2n$$

$$\implies \begin{bmatrix} \exp(-n\xi_0) & \cdots & \exp(0\xi_0) & \cdots & \exp(n\xi_0) \\ \exp(-n\xi_1) & \cdots & \exp(0\xi_1) & \cdots & \exp(n\xi_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \exp(-n\xi_{2n-1}) & \cdots & \exp(0\xi_{2n-1}) & \cdots & \exp(n\xi_{2n-1}) \\ \exp(-n\xi_{2n}) & \cdots & \exp(0\xi_{2n}) & \cdots & \exp(n\xi_{2n}) \end{bmatrix} \begin{bmatrix} c_{-n} \\ \vdots \\ c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_2) \\ \vdots \\ f(x_{2n-1}) \\ f(x_{2n}) \end{bmatrix} \quad (3)$$



(C) Solve linear system for the coefficients $c_{-n}, \ldots, c_n$

(D) We now have our interpolant, $f_a(x)$!

Notes.

- Use ***uniformly spaced*** interpolation points, and exclude right boundary point.
- Don't actually construct and solve (3)

Use the Fast Fourier Transform. Instead of getting coefficients in $O(n^2)$ operations, does it in $O(n\log(n))$ operations. ***HUGE*** savings when n is large.

How to perform trigonometric interpolation on a computer

Recap. So how do we compute the approximation $f_a(x)$ associated with the $\mathcal{T}^n[a, b]$ subspace?

(A) For an $f(x)$ sampled at $2n + 1$ evenly spaced interpolation points & assembled into a vector $\mathbf{f}$…

(B) Extract the coefficients via

$$\mathbf{c} = \frac{1}{2n+1} fft(\mathbf{f})$$

Note: Python returns the coefficients in the order

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \\ c_{-n} \\ \vdots \\ c_{-1} \end{bmatrix}$$

(C) We now have $f_a(x)$ via

$$f_a(x) = \sum_{k=-n}^{n} c_k \exp\left(\frac{2\pi i k(x-a)}{(b-a)}\right)$$