Lecture 22: FEM for IBVPs

Today:

- ***Finite element methods*** applied to initial boundary value problems (IBVPs)

Where are we up to now?

Last week.

(A) We developed ***finite difference methods*** methods for solving ***initial boundary value problems.*** Our approach first spatially discretized the IBVP which resulted in an IVP. We then solved the IVP using previously discussed time stepping methods.

This week. FEM for IBVPs

- We still use the method of lines to spatially discretize our PDE and solve the resulting IVP with a time stepping method
- This time we will spatially discretize using a ***local spectral method*** rather than local polynomial interpolation
- We can then solve the resulting IVP using previously discussed time stepping methods

Our canonical IBVP

We will again consider the 1D ***heat equation*** for our derivations

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + g(x, t), \quad 0 \le t \le T,\ a \le x \le b \tag{1}$$

where κ is the heat diffusivity and $g(x, t)$ is a prescribed forcing.

We will assume zero Dirichlet ***boundary conditions***

$$u(x, t = 0) = \eta(x)$$

Prescribed initial condition; $\eta(x)$ is given

$$u(x = a, t) = g_a(t) = 0$$

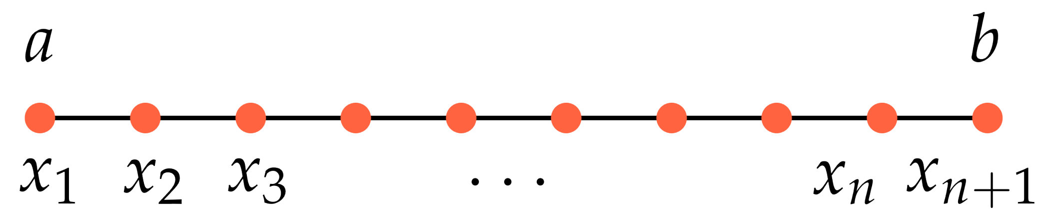
$$u(x = b, t) = g_b(t) = 0$$

Prescribed boundary condition

Previously: spatial discretization with ***local polynomial interpolation***

We used a uniformly distributed set of points:

$$x_j = a + \frac{(b-a)(j-1)}{n}, \quad j = 1, \ldots, n+1$$



With finite difference methods, we approximated the solution $u(x,t)$ with local polynomial interpolation:

$$u(x,t) \approx \sum_{i=j-p/2}^{j+p/2} u_i(t) L_i^{(j)}(x) \qquad (2)$$

which resulted in the following expression for our PDE:

$$\implies \dot{u}_j(t) = \kappa \sum_{i=j-p/2}^{j+p/2} u_i(t) \left. \frac{d^2 L_i^{(j)}}{dx^2} \right|_{x=x_j} + g(x_j, t) \qquad (j = 2, \ldots, n)$$

For p=2, this term simplified to… $\frac{1}{\Delta x^2}(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t))$

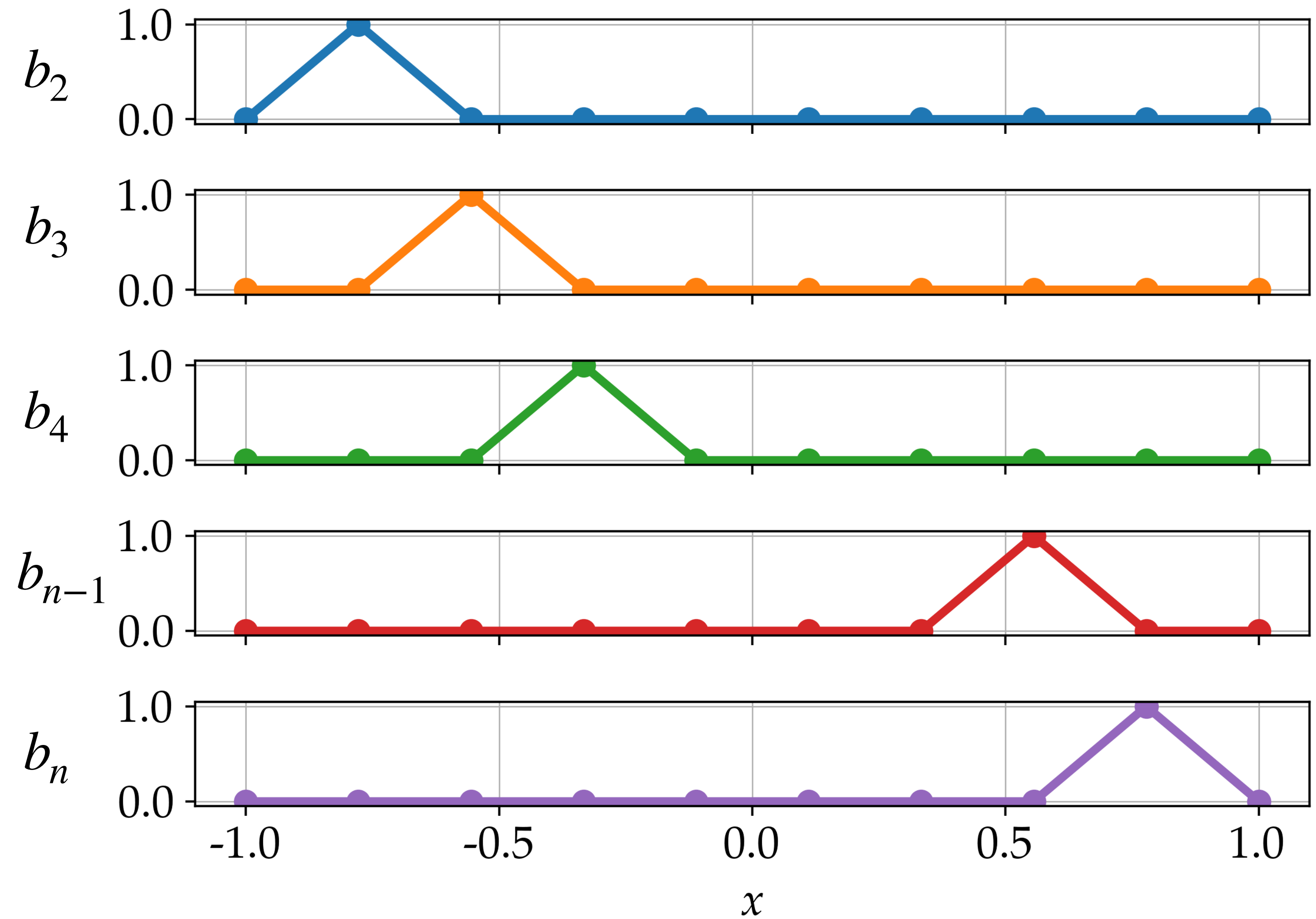
which results in a matrix system with a typical IVP structure.

This time: spatial discretization with ***local spectral methods***

This time, we will approximate the solution $u_a(x,t)$ on the subspace of piecewise-linear functions $\mathscr{V}_n^L$ such that:

$$\hat{u}(x,t) = \sum_{i=2}^{n} c_i(t) b_i(x)$$

We assume time dependence is captured by coefficients c_i and spatial dependence is captured by basis functions b_i



Recall, any piecewise linear function $q(x)$ can be written in terms of the basis functions and its *nodal values* $q(x_1), q(x_2), \ldots, q(x_n)$ as $q(x) = \sum_{i=2}^{n} q(x_i) b_i(x)$, so…

$$\hat{u}(x,t) = \sum_{j=2}^{n} u_i(t) b_i(x) \qquad (3)$$

$$b_i(x) = \begin{cases} \frac{1}{\Delta x}[x - a - (i-2)\Delta x], & \text{if } x \in [x_{i-1}, x_i] \\ -\frac{1}{\Delta x}[x - a - i\Delta x], & \text{if } x \in [x_i, x_{i+1}] \\ 0, & \text{else} \end{cases} \qquad i = 2, \ldots, n$$

How do we get our approximation?

$$\hat{u}(x,t) = \sum_{j=2}^{n} u_i(t) b_i(x)$$

How do we determine our coefficients $u_i(t)$?

Recall that spectral methods for BVPs minimize the overall (least-squares) error:

$$\hat{u}(x,t) = \min_{\hat{u} \in \mathcal{V}_n^L} ||u - \hat{u}||^2 \qquad (3)$$

The $\hat{u}(x,t)$ which satisfies (3) satisfies the following for some inner product $(\cdot, \cdot)$

$$(u - \hat{u}, b_j) = 0, \quad j = 2, \ldots, n \qquad (4)$$

Based on (4), the optimal $\hat{u}(x,t)$ then satisfies the following for the ***1D Poisson equation BVPs***

$$(u - \hat{u}, b_j) = 0, \quad j = 2, \ldots, n \qquad (4)$$

$$\begin{bmatrix} (b_0, b_0) & (b_1, b_0) & \cdots & (b_{n-1}, b_0) & (b_n, b_0) \\ (b_0, b_1) & (b_1, b_1) & \cdots & (b_{n-1}, b_1) & (b_n, b_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (b_0, b_{n-1}) & (b_1, b_{n-1}) & \cdots & (b_{n-1}, b_{n-1}) & (b_n, b_{n-1}) \\ (b_0, b_n) & (b_1, b_n) & \cdots & (b_{n-1}, b_n) & (b_n, b_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} (f, b_0) \\ (f, b_1) \\ \vdots \\ (f, b_{n-1}) \\ (f, b_n) \end{bmatrix}$$

Which reduces to the following if we assume a linear-piecewise subspace (and corresponding basis functions

$$\frac{1}{\Delta x} \begin{bmatrix} -2 & 1 & \cdots & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} \int_a^{a+2\Delta x} f(x) b_2(x) dx \\ \int_{a+\Delta x}^{a+3\Delta x} f(x) b_3(x) dx \\ \vdots \\ \int_{a+(n-2)\Delta x}^{a+(n)\Delta x} f(x) b_{n-1}(x) dx \\ \int_{a+(n-1)\Delta x}^{a+(n+1)\Delta x} f(x) b_n(x) dx \end{bmatrix}$$

What changes for IBVPs?

When we move from BVPs to IBVPs, two questions now arise:

(1) What is the appropriate inner product to use?

(2) How does the resulting matrix system change for our PDE (rather than the ODE we solved in BVPs)?

(1) What is the appropriate inner product to use?

The heat equation we are solving is just an unsteady version of the Poisson equation, so the same notion of energy product applies!

$$(f, g)_E = \kappa \int_a^b f'(x,t) g'(x,t) dx, \quad \forall f, g \in \mathscr{V}_n^L$$

Partial derivatives with respect to x

How does the matrix system change?

$$\hat{u}(x,t) = \sum_{j=2}^{n} u_i(t)\phi_i(x) \qquad (3)$$

Assumed approximation

$$(u - \hat{u}, \phi_j)_E = 0, \quad j = 2,\ldots,n \qquad (4)$$

Least-squares condition

$$(u - \sum_{i=2}^{n} u_i\phi_i, \phi_j)_E = 0, \quad j = 2,\ldots,n \qquad (5)$$

$$(\sum_{i=2}^{n} u_i\phi_i, \phi_j)_E = (u, \phi_j)_E, \quad j = 2,\ldots,n \qquad (6)$$

$$\sum_{i=2}^{n} u_i(\phi_i, \phi_j)_E = (u, \phi_j)_E, \quad j = 2,\ldots,n \qquad (7)$$

Let's expand the RHS of (7)

$$(u,\phi_j)_E = \kappa \int_a^b u'(x,t)\phi_j'(x)dx \tag{8}$$

$$= \kappa[u'(x,t)\phi_j(x)]_a^b - \kappa \int_a^b u''(x,t)\phi_j(x)dx \quad \text{[Integrate by parts]} \tag{9}$$

$$= -\kappa \int_a^b u''(x,t)\phi_j(x)dx \quad \left[\phi_j \in \mathcal{V}_n^L \text{ so it satisfies the BCs}\right] \tag{10}$$

$$= \int_a^b \left[g(x,t) - \frac{\partial u}{\partial t}\right]\phi_j(x)dx \quad [u(x,t) \text{ satisfies the heat equation}] \tag{11}$$

$$= \int_a^b g(x,t)\phi_j(x)dx - \int_a^b \frac{\partial u}{\partial t}\phi_j(x)dx \tag{12}$$

This looks promising EXCEPT for the u term in the RHS

But we actually have an assumed approximation for u!

$$(u,\phi_j)_E = \int_a^b g(x,t)\phi_j(x)dx - \int_a^b \frac{\partial}{\partial t}\left[\sum_{i=2}^{n} u_i(t)\phi_i(x)\right]\phi_j(x)dx \quad (13)$$

$$= \int_a^b g(x,t)\phi_j(x)dx - \frac{\partial}{\partial t}\left[\sum_{i=2}^{n} u_i(t)\right]\int_a^b \phi_i(x)\phi_j(x)dx \quad (14)$$

$$= \int_a^b g(x,t)\phi_j(x)dx - \sum_{i=2}^{n} \dot{u}_i(t)\int_a^b \phi_i(x)\phi_j(x)dx \quad (15)$$

Plugging back into (7)…

$$\sum_{i=2}^{n} u_i\,(\phi_i,\phi_j)_E = \int_a^b g(x,t)\phi_j(x)dx - \sum_{i=2}^{n} \dot{u}_i(t)\int_a^b \phi_i(x)\phi_j(x)dx, \quad j=2,\ldots,n \quad (17)$$

$$= (g,\phi_j)_s - \sum_{i=2}^{n} \dot{u}_i(t)\,(\phi_i,\phi_j)_s \quad (18)$$

Re-arranging…

$$\sum_{i=2}^{n} \dot{u}_i(t)(\phi_i, \phi_j)_s = -\sum_{i=2}^{n} u_i(\phi_i, \phi_j)_E + (g, \phi_j)_s, \quad j = 2, \ldots, n$$

Then in matrix form we get the following…

$$\begin{bmatrix} (\phi_2,\phi_2)_s & (\phi_3,\phi_2)_s & \cdots & (\phi_{n-1},\phi_2)_s & (\phi_n,\phi_2)_s \\ (\phi_2,\phi_3)_s & (\phi_3,\phi_3)_s & \cdots & (\phi_{n-1},\phi_3)_s & (\phi_n,\phi_3)_s \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (\phi_2,\phi_{n-1})_s & (\phi_3,\phi_{n-1})_s & \cdots & (\phi_{n-1},\phi_{n-1})_s & (\phi_n,\phi_{n-1})_s \\ (\phi_2,\phi_n)_s & (\phi_3,\phi_n)_s & \cdots & (\phi_{n-1},\phi_n)_s & (\phi_n,\phi_n)_s \end{bmatrix} \dot{\begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}} =$$

$$-\begin{bmatrix} (\phi_2,\phi_2)_E & (\phi_3,\phi_2)_E & \cdots & (\phi_{n-1},\phi_2)_E & (\phi_n,\phi_2)_E \\ (\phi_2,\phi_3)_E & (\phi_3,\phi_3)_E & \cdots & (\phi_{n-1},\phi_3)_E & (\phi_n,\phi_3)_E \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (\phi_2,\phi_{n-1})_E & (\phi_3,\phi_{n-1})_E & \cdots & (\phi_{n-1},\phi_{n-1})_E & (\phi_n,\phi_{n-1})_E \\ (\phi_2,\phi_n)_E & (\phi_3,\phi_n)_E & \cdots & (\phi_{n-1},\phi_n)_E & (\phi_n,\phi_n)_E \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} + \begin{bmatrix} (g,\phi_2)_s \\ (g,\phi_3)_s \\ \vdots \\ (g,\phi_{n-1})_s \\ (g,\phi_n)_s \end{bmatrix}$$

Which we can write as… $\boldsymbol{M}\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + g(t)$ (19)

But we want an IVP that looks like $\dot{\boldsymbol{u}} = f(\boldsymbol{u}, t)$

Re-arrange to…

$$\boldsymbol{M}\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + g(t)$$

$$\dot{\boldsymbol{u}} = \boldsymbol{M}^{-1}(\boldsymbol{A}\boldsymbol{u} + g(t))$$

$$\dot{\boldsymbol{u}} = f(\boldsymbol{u}, t)$$

Use your favorite time stepping method to solve this IVP!

What is $\boldsymbol{A}$? From FEMs for BVPs, we know that…

$$\phi_i'(x) = \begin{cases} \frac{1}{\Delta x} & x \in [x_{i-1}, x_i] \\ -\frac{1}{\Delta x} & x \in [x_i, x_{i+1}] \\ 0 & else \end{cases}$$

$$(\phi_i, \phi_{i-1})_z = \int_{a+(i-1)\Delta x}^{a+i\Delta x} \left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = -\frac{1}{\Delta x}$$

$$(\phi_i, \phi_i)_E = \int_{a+(i-1)\Delta x}^{a+(i+1)\Delta x} \left(\frac{1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = \frac{2}{\Delta x}$$

$$(\phi_i, \phi_{i+1})_E = \int_{a+i\Delta x}^{a+(i+1)\Delta x} \left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx = -\frac{1}{\Delta x}$$

$$\boldsymbol{A} = \frac{\kappa}{\Delta x}\begin{bmatrix} -2 & 1 & \cdots & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

What about $\boldsymbol{M}$?

Our only non-zero terms will be $(\phi_{i-1}, \phi_i)_s, \quad (\phi_i, \phi_i)_s, \quad (\phi_i, \phi_{i+1})_s$

These terms end up being…

$$(\phi_{i-1}, \phi_i)_s = \int_{a+(i-1)\Delta x}^{a+i\Delta x} \phi_{i-1}(x)\phi_i(x)dx = \frac{1}{6}\Delta x$$

Using Mathematica…

$$(\phi_i, \phi_i)_s = \int_{a+(i-1)\Delta x}^{a+(i+1)\Delta x} \phi_i(x)\phi_i(x)dx = \frac{2}{3}\Delta x$$

$$(\phi_i, \phi_{i+1})_s = \int_{a+i\Delta x}^{a+(i+1)\Delta x} \phi_i(x)\phi_{i+1}(x)dx = \frac{1}{6}\Delta x$$

Which gives the following for $\boldsymbol{M}$…

$$\boldsymbol{M} = \frac{\Delta x}{6}\begin{bmatrix} 4 & 1 & \cdots & 0 & 0 \\ 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{bmatrix}$$

Solve the resulting IVP!

We have now re-formulated our IBVP, defined by

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + g(x,t), \quad 0 \leq t \leq T,\ a \leq x \leq b$$

$$u(x, t=0) = \eta(x)$$

$$u(x=a, t) = g_a(t) = 0$$

$$u(x=b, t) = g_b(t) = 0$$

as an IVP, defined by

$$\dot{\boldsymbol{u}} = f(\boldsymbol{u}, t) = \boldsymbol{M}^{-1}(\boldsymbol{A}\boldsymbol{u} + g(t))$$

which we can solve with any finite difference method!

Are there certain classes of FD methods that are better for the heat equation though?

Some important facts

Similar to FD methods for IBVPs, but here are some important facts about FEM for IBVPs:

Using method of lines to (i) discretize in space via an FEM method and (ii) solve the resulting IVP with a FD method gives ***convergent solutions***

The numerical solution will approach the true solution as $\Delta x \to 0$, $\Delta t \to 0$

The ***spatial convergence rate*** of this method is equal to the order of the spatial discretization error.

E.g., for the second order central difference method, the error in space is $O(\Delta x^2)$

The ***temporal convergence rate*** of this method is equal to the order of the temporal discretization error.

E.g., using the trapezoid method to solve the spatially discrete IVP, the error in time is $O(\Delta t^2)$