Lecture 21: IBVPs (2)

Today:

- ***Finite difference methods*** applied to ***initial boundary value problems (IBVPs)***
  - Leverage the spatially discretized initial value problem to advance the solution in time

Where are we up to now?

Previously. Finite difference methods for IBVPs — PDEs that depend on time and space

- Use an FD method to ***discretize*** the PDE in space → gives an IVP
- Solve the resulting IBVP with an FD method

This approach is called the ***method of lines***

$$\begin{bmatrix} \dot{u}_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \frac{\kappa}{\Delta x^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} + \begin{bmatrix} g(x_2, t) + \frac{\kappa g_a(t)}{\Delta x^2} \\ g(x_3, t) \\ \vdots \\ g(x_{n-1}, t) \\ g(x_n, t) + \frac{\kappa g_b(t)}{\Delta x^2} \end{bmatrix}$$

Can write more succinctly as $\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{g}$

Today.

- Details on solving the spatially discrete IVP
- Seeing an example

Solving the IVP using explicit methods

We can write the spatially discrete IVP as $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t)$, where $\mathbf{f}(\mathbf{u}, t) = \mathbf{A}\mathbf{u} + \mathbf{g}(t)$

If we were to apply forward Euler:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{f}(\mathbf{u}_k, t_k)$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t(\mathbf{A}\mathbf{u}_k + \mathbf{g}(t_k))$$

Start with $\mathbf{u}_0$ given by the initial condition and can advance from there!

Or for AB2:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \frac{\Delta t}{2}\left[-\mathbf{f}(\mathbf{u}_{k-1}, t_{k-1}) + 3\mathbf{f}(\mathbf{u}_k, t_k)\right]$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \frac{\Delta t}{2}\left[-\left(\mathbf{A}\mathbf{u}_{k-1} + \mathbf{g}(t_{k-1})\right) + 3\left(\mathbf{A}\mathbf{u}_k + \mathbf{g}(t_k)\right)\right]$$

Start with $\mathbf{u}_0$ given by the initial condition and get $\mathbf{u}_1$ using a one-step method like forward Euler or Heun's.

Can advance from there!

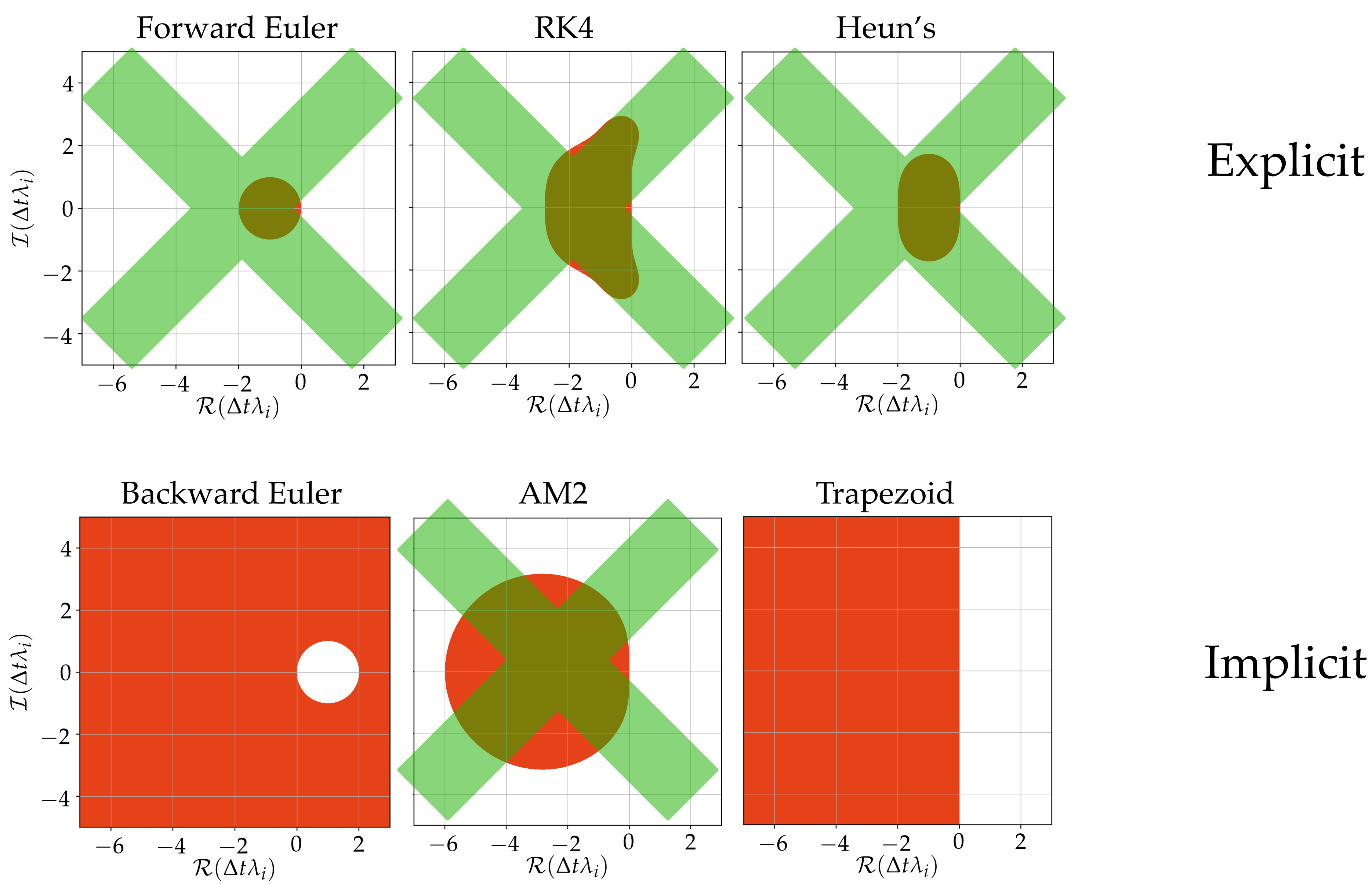
Stability restrictions for the heat equation

It turns out that the heat equation is a ***stiff IVP*** $\Longrightarrow$ imposes severe stability restrictions

Unlike the N body problem, where you could use explicit methods without issue!

Explicit methods are therefore ***not*** well suited to solving the spatially discretized heat equation. Want a method with a big stability region…

Stability regions too small $\Longrightarrow$ need tiny Δt for stable sims



Solving the IVP using backward Euler

Let's see how to apply a better suited time stepping method to the spatially discrete heat equation. Start with BE…

We can write the spatially discrete IVP as $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t)$, where $\mathbf{f}(\mathbf{u}, t) = \mathbf{A}\mathbf{u} + \mathbf{g}(t)$

If we were to apply backward Euler:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{f}(\mathbf{u}_{k+1}, t_{k+1})$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t(\mathbf{A}\mathbf{u}_{k+1} + \mathbf{g}(t_{k+1}))$$

We can rearrange this to get an equation for $\mathbf{u}_{k+1}$:

$$\mathbf{u}_{k+1} - \Delta t \mathbf{A}\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{g}(t_{k+1})$$

$$\implies (\mathbf{I} - \Delta t \mathbf{A})\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{g}(t_{k+1})$$

$$\implies \mathbf{u}_{k+1} = (\mathbf{I} - \Delta t \mathbf{A})^{-1}(\mathbf{u}_k + \Delta t \mathbf{g}(t_{k+1}))$$

Start with $\mathbf{u}_0$ and can advance from there!

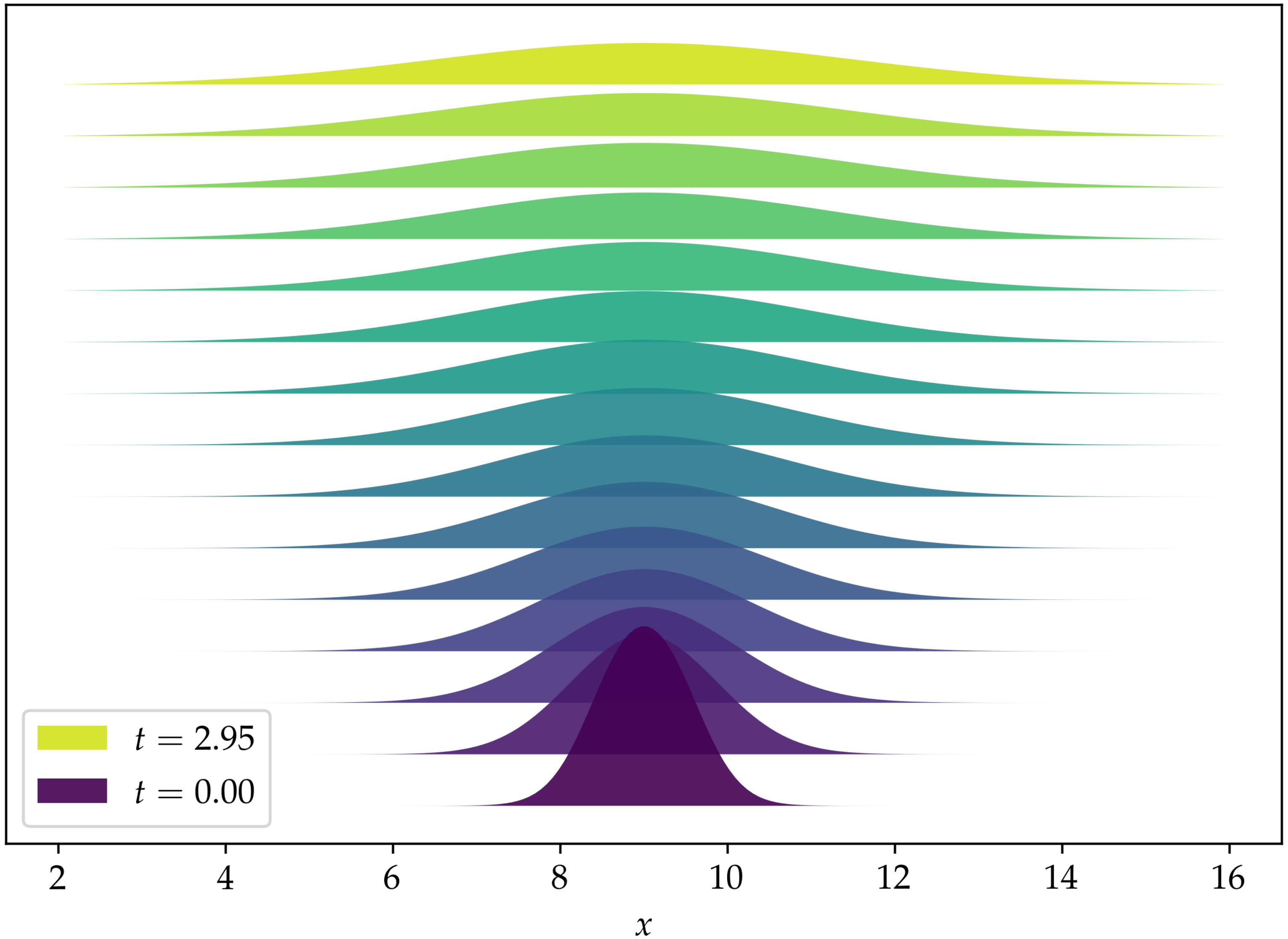
What would the equations look like for the trapezoid method?

Results for an example problem

Consider $a = 2$, $b = 16$, $\kappa = 1$, $g(x,t) = 0$, $g_a(t) = 0$, $g_b(t) = 0$, and

$$\eta(x) = \exp\left[-\left(\frac{x - \frac{a+b}{2}}{2\sigma}\right)^2\right] \qquad \text{where } \sigma = 0.3$$

Simulating using our 2nd order central difference scheme for spatial discretization and the trapezoid method, with $\Delta t = \Delta x \approx 0.07$, gives



Some important facts

We don't have time to prove, but here are some important facts:

Using method of lines to (i) discretize in space via a FD method and (ii) solve the resulting IVP with a FD method gives ***convergent solutions***

The numerical solution will approach the true solution as $\Delta x \to 0$, $\Delta t \to 0$

The ***spatial convergence rate*** of this method is equal to the order of the spatial discretization error.

E.g., for the second order central difference method, the error in space is $O(\Delta x^2)$

The ***temporal convergence rate*** of this method is equal to the order of the temporal discretization error.

E.g., using the trapezoid method to solve the spatially discrete IVP, the error in time is $O(\Delta t^2)$

While we unfortunately do not have time to go through the details of these convergence properties, a very brief overview is that convergence depends on (you guessed it!) i) using Taylor series of the truncation error and ii) combining this analysis with a notion of stability.