Lecture 2: Vector Spaces

Today:

- Introduce the problem of **approximating functions**
- Introduce key concepts — **vector space, subspace**

Function approximation

Recall our roadmap…

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \quad (1)$$

Let's make some observations:

Start w/ this today!

$\mathbf{u}$ is a function of $\mathbf{x}, t$ (space and time) → **Step 1**: we will learn how to approximate *prescribed* functions

Equation (1) is a PDE that depends on both space and time → **Step 2**: we will learn how to numerically solve *ODEs* in *time*

$$\frac{\partial \mathbf{u}}{\partial t} = f(\mathbf{u}, t) \approx f_a(\mathbf{u}, t)$$

→ **Step 3**: we will learn how to numerically solve *ODEs* in *space*

We can then use $f_a(\mathbf{u}, t)$ to solve an ODE in time

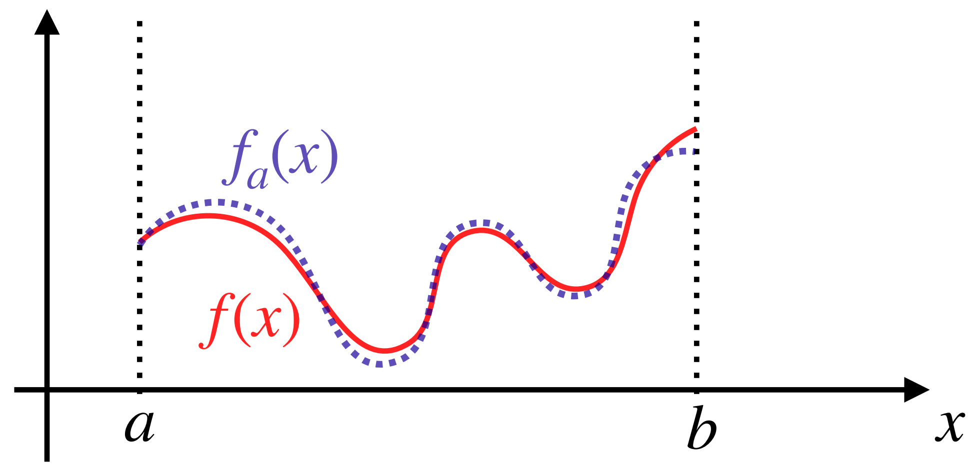
→ **Step 4**: we will learn how to numerically solve *PDEs* in *space and time*

This is our roadmap for the semester!

Goal of function approximation

Goal: find a function, $f_a(x)$, that approximates a given function, $f(x)$, *accurately* on $x \in [a, b]$

x belonging to the interval $[a, b]$



How do we address these challenges?

What are the key ***challenges*** to this approach?

- There are infinitely many possible functions! Can't handle on a computer
- What does accurately mean?

Introduce key concept of ***norm*** to quantify the *size of the error*

Restrict the set of possible functions we are considering
Introduce key concepts of a ***vector space & subspace***

Vector spaces and subspaces

Vector spaces

Definition: A <u>vector space</u> is a set of elements where addition and scalar multiplication are well defined

Ex: $C[a, b]$ ← The space of all continuous functions defined on the interval $[a, b]$

Let's check that this is a vector space:

(1) *Additivity*:

- consider two continuous functions, $f(x), g(x)$
- $f(x) + g(x)$ makes sense (can add continuous functions)

(2) *Scalar multiplication*:

- consider a function $f(x)$ and a scalar number $\alpha \in \mathbb{R}$ ← α belongs to the set of real numbers
- $\alpha f(x)$ makes sense (can multiply continuous functions by a scalar number)

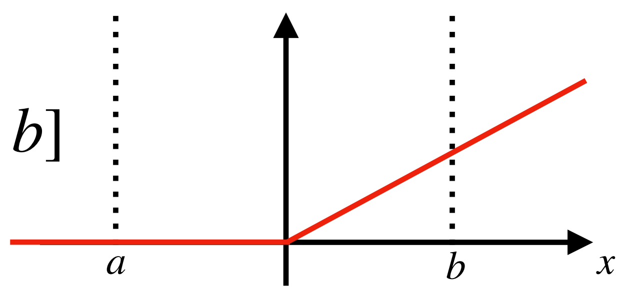
Activity:
Playing with vector spaces!

Activity:

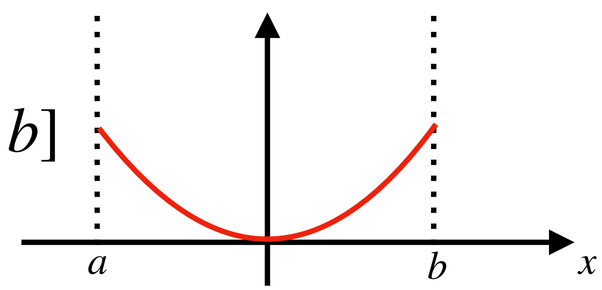
(A) Is $C^k[a,b]$ (the space of k-times differential functions on the interval $[a,b]$) a vector space?

(B) What about $\mathscr{P}^n[a,b]$ (the space of all polynomials of degree n or less on the interval $[a,b]$)?

$C^0[a,b]$



$\mathscr{P}^2[a,b]$



Key takeaway: How did we know the answers to (1) are vector spaces??

Check that additivity and scalar multiplication makes sense

Some questions you may have right now.

A) Why are vector spaces important for approximating functions?

B) Why is it useful to approximate the desired function, $f(x)$, with another function, $f_a(x)$?

Subspaces

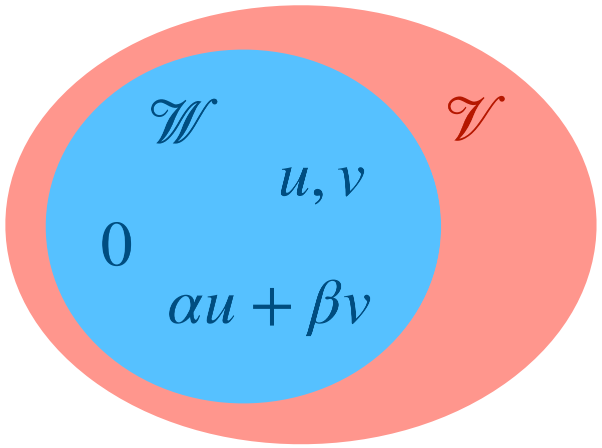
Don't forget about this!

Definition: A subset $\mathscr{W}$ of vector space $\mathscr{V}$ is called a *subspace* of $\mathscr{V}$ if:

(1) $0 \in \mathscr{W}$ ← The 0 function, $f(x) = 0$, belongs to $\mathscr{W}$

(2) $\alpha u + \beta v \in \mathscr{W}$ for $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathscr{W}$

Elements of the subspace stay in the subspace after basic arithmetic operations



Example: $\mathscr{W} = C^1[a,b]$ is a *subspace* of $\mathscr{V} = C[a,b]$

(1) $\mathscr{W}$ is a subset of $\mathscr{V}$

(2) $0 \in C^1[a,b]$

(3) Multiply two differentiable functions by a scalar and add them together $\Longrightarrow$ get a differentiable function

Activity:
Playing with subspaces!

Activity:

(A) Is $C^k[a, b]$ (the space of k-times differential functions on the interval $[a, b]$) a subspace of $C[a, b]$?

(B) $\mathscr{P}^n[a, b]$ (the space of all polynomials of degree n or less on the interval $[a, b]$) is a subspace of $C[a, b]$?

Some questions you may have now:

(A) How do subspaces help with approximating functions?

(B) Why is $C[a, b]$ not a subspace of $C^1[a, b]$ if $C[a, b]$ satisfies definitions (1) & (2) of a subspace?

Vector spaces and subspaces: Why do we care?

Motivate by example:

(1) Let's say we know $f \in C[a, b]$

(2) I can pick a subspace $\mathscr{P}^n[a, b]$

(3) *Any* degree-n polynomial can be written as

$$f_a(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

(4) Turns the ***ambiguous goal*** of "approximate one of infinitely many possible $f(x)$" into the ***concrete, finite dimensional aim*** "solve for $n + 1$ coefficients $c_0, \ldots, c_n$"

So we care because…

vector space: useful concept because it helps us

(1) characterize the function we want to approximate

(2) decide which subspace to use to build our approximating function

subspace: useful concept because it helps us

(1) limit the infinitely many possible functions to a finite-sized subset