Global spectral methods for BVPs
Part 2: Some examples

In the last lecture, we developed a framework for approximating the solution to a BVP using a global spectral method. That is, we represented the solution as a linear combination of global basis functions, and developed an equation for the coefficients by recognizing that the optimal approximation satisfies the very special property that it produces an error that is orthogonal to the approximation subspace.

In this lecture, we will apply this framework to a few examples.

1 *Global spectral methods with polynomials*

In this section, we consider using globally defined polynomials to develop a spectral method for the 1D Poisson equation. That is, we want an approximation space defined by n^{th} order polynomials. This suggests that we use our handy Lagrange polynomials $\{L_1, \ldots, L_{n+1}\}$. But we have one more constraint: this approximation space must automatically satisfy the boundary conditions. Said differently, we want to approximate $\hat{u}$ onto the space $\mathcal{P}_0^n$, not $\mathcal{P}^n$.

Notice the distinction in notation: whereas $\mathcal{P}^n$ is the space defined by any and all degree-n polynomials, $\mathcal{P}_0^n$ is defined by any degree-n polynomial that satisfies the zero Dirichlet boundary conditions at $x = a$ and $x = b$.

How do we modify our basis functions to incorporate this boundary condition? The answer to this question comes from the observation that, by construction, $L_i(x_j)$ is 1 if $x = j$ and 0 otherwise. Thus, $\{L_2, \ldots, L_n\}$ are all zero by default at $x = a$ and $x = b$. By contrast, L_1 is nonzero at $x = a$ and L_n is nonzero at $x = b$, and will never satisfy the boundary conditions. So a suitable basis for $\mathcal{P}_0^n$ is $\{L_2, \ldots, L_n\}$. The indices i_1 and i_n are therefore $i_1 = 2$, $i_n = n$.

It may seem odd that there are two fewer basis functions associated with $\mathcal{P}_0^n$ compared with $\mathcal{P}^n$. In fact, there is a very good reason for this: the two boundary conditions remove two degrees of freedom on the solution (it restricts what values u can take on at two points). We are therefore only *allowed* to use n-1 basis functions.

Now that we have picked our boundary conditions, we can write our approximate solution $\hat{u}$ as

$$u(x) \approx \hat{u}(x) = \sum_{k=2}^{n} c_k L_k(x) \tag{1}$$

Thus, using this basis in (17) from the last lecture, we have an equation that we can solve for our coefficients:

$$\begin{bmatrix} (L_2, L_2)_E & (L_3, L_2)_E & \cdots & (L_{n-1}, L_2)_E & (L_n, L_2)_E \\ (L_2, L_3)_E & (L_3, L_3)_E & \cdots & (L_{n-1}, L_3)_E & (L_n, L_3)_E \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (L_2, L_{n-1})_E & (L_3, L_{n-1})_E & \cdots & (L_{n-1}, L_{n-1})_E & (L_n, L_{n-1})_E \\ (L_2, L_n)_E & (L_3, L_n)_E & \cdots & (L_{n-1}, L_n)_E & (L_n, L_n)_E \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} -(f, L_2)_s \\ -(f, L_3)_s \\ \vdots \\ -(f, L_{n-1})_s \\ -(f, L_n)_s \end{bmatrix} \tag{2}$$

To solve (2), we need a way to evaluate the energy inner products in the matrix and the standard inner products in the righthand side term. How do we do this? Well the matrix terms involve polynomial

Note that, just as we saw in least squares function approximation for the standard inner product, the Lagrange polynomials are not orthogonal with respect to the energy inner product. But if we wanted to be fancy, we could construct a set of orthogonal polynomials using the Gram-Schmidt orthogonalization process, as you did in a previous homework. This would have the ***huge*** benefit of making the matrix in the linear system 2 the identity matrix, which makes the linear system trivially solvable.

basis functions and can be evaluated analytically using symbolic toolboxes. For the righthand side term, these analytical computations may not be tractable depending on f. But that is not a problem; we can handle these terms using our favorite quadrature rule!

2 *Global spectral methods with trigonometric functions*

Now we use a spectral method derived from trigonometric functions to solve the 1D Poisson equation. Just as we did for polynomials, we must ensure that our space satisfies the zero boundary conditions; we are therefore in pursuit of a space $\mathcal{T}_0^n$ (and not the space $\mathcal{T}^n$ that we used for trigonometric interpolation).

What is a basis for this space? Well recall that a basis for the larger space $\mathcal{T}^n$ was comprised of sine and cosine functions that were integer periodic on $[a, b]$. However, each of these cosine functions is nonzero at $x = a$ and $x = b$, and consequently does not satisfy the boundary conditions. We thus construct our basis for $\mathcal{T}_0^n$ using sine functions. In particular, they must be shifted to be zero at $x = a$, so that the basis is

$$\left\{ \sin\left(\frac{\pi(x-a)}{b-a}\right), \sin\left(\frac{2\pi(x-a)}{b-a}\right), \ldots, \sin\left(\frac{n\pi(x-a)}{b-a}\right) \right\} \tag{3}$$

The indices i_1 and i_n are therefore $i_1 = 1$, $i_n = n$.

Using our basis for $\mathcal{T}_0^n$, we can write our approximation $\hat{u}$ as

$$u(x) \approx \hat{u}(x) = \sum_{k=1}^{n} c_k \sin\left(\frac{k\pi(x-a)}{b-a}\right) \tag{4}$$

We again use (17) from the last lecture to solve for the unknown coefficients. Before, writing out the matrix system, however, let us notice something beautiful about the sine basis. Taking the energy inner product between two of the sine basis functions gives

$$\begin{aligned}
&\left(\sin\left(\frac{j\pi(x-a)}{b-a}\right), \sin\left(\frac{k\pi(x-a)}{b-a}\right)\right)_E \\
&\quad = \int_a^b \sin'\left(\frac{j\pi(x-a)}{b-a}\right) \sin'\left(\frac{k\pi(x-a)}{b-a}\right) dx \\
&\quad = \frac{jk\pi^2}{(b-a)^2} \int_a^b \cos\left(\frac{j\pi(x-a)}{b-a}\right) \cos\left(\frac{k\pi(x-a)}{b-a}\right) \\
&\quad = \begin{cases} 0 & j \neq k \\ \frac{j^2\pi^2}{2(b-a)} & j = k \end{cases}
\end{aligned} \tag{5}$$

That is, the sine basis is orthogonal with respect to the energy inner product.

Because of this property of the sine basis functions, the system (17)

from the last lecture for obtaining the coefficients simplifies to

$$-\frac{\pi^2}{2(b-a)}\begin{bmatrix} 1^2 & 0 & \cdots & 0 & 0 \\ 0 & 2^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-1)^2 & 0 \\ 0 & 0 & \cdots & 0 & n^2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \left(f, \sin\left(\frac{\pi(x-a)}{b-a}\right)\right)_s \\ \left(f, \sin\left(\frac{2\pi(x-a)}{b-a}\right)\right)_s \\ \vdots \\ \left(f, \sin\left(\frac{(n-1)\pi(x-a)}{b-a}\right)\right)_s \\ \left(f, \sin\left(\frac{n\pi(x-a)}{b-a}\right)\right)_s \end{bmatrix} \tag{6}$$

We can therefore solve for the coefficients directly as

$$c_j = -\frac{2(b-a)}{j^2\pi^2}\left(f(x), \sin\left(\frac{j\pi(x-a)}{b-a}\right)\right)_s, \quad j = 1, \ldots, n \tag{7}$$

Just as we observed for the spectral method involving global polynomials, (7) involves the standard inner product that we can evaluate using one of the quadrature rules we developed earlier in class. A trapezoidal rule is one option, but approaches based on global interpolation will enable greater accuracy with a much smaller number of points.