Lecture 16: Spectral Methods for BVPs (1)

Today:

- Continue with ***boundary value problems (BVPs)***
  - This time: don't use finite difference methods.
  - Use ***global spectral methods***
    - Analog of least squares function approximation!

Where are we up to now?

Previously.

(A) We learned how to approximate the solution of a BVP using an FD method

(B) We learned how to predict the error of that FD method using truncation error and stability

Today. Solutions to BVPs using ***global spectral methods***

Philosophy behind spectral methods

Whereas finite difference methods are based on locally interpolating the differential equation, spectral methods minimize the least-squares error in the numerical solution.

This approach is an analog of least squares function approximation (with a prescribed function)!

Because of this strong connection, we will review least-squares function approximation and leverage this to develop global spectral methods

Reminder: least squares function approximation

Goal: find a function, $f_a(x)$, that approximates a given function, $f(x)$, *accurately* on $x \in [a, b]$

(A) Pick a ***subspace*** that we want to approximate onto (e.g., $\mathscr{P}^n[a, b]$ or $\mathscr{T}^n[a, b]$)

(B) Write $f_a(x)$, as a linear combination of the basis functions in terms of unknown coefficients

$$f_a(x) = \sum_{j=0}^{n} c_j b_j(x) \qquad (1)$$

(C) Need n equations to solve for the n unknown coefficients. Seek to ***minimize the overall error***. That is, we will look for an $f_a(x)$ that satisfies

$$\min_{f_a \in \mathscr{V}} ||f - f_a||^2 \qquad (2)$$

(D) Notice that the optimal f_a satisfies

$$(f - f_a, b_i) = 0, \quad i = 0,\ldots,n \qquad (3)$$

$$\implies \begin{bmatrix} (b_0, b_0) & (b_1, b_0) & \cdots & (b_{n-1}, b_0) & (b_n, b_0) \\ (b_0, b_1) & (b_1, b_1) & \cdots & (b_{n-1}, b_1) & (b_n, b_1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (b_0, b_{n-1}) & (b_1, b_{n-1}) & \cdots & (b_{n-1}, b_{n-1}) & (b_n, b_{n-1}) \\ (b_0, b_n) & (b_1, b_n) & \cdots & (b_{n-1}, b_n) & (b_n, b_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} (f, b_0) \\ (f, b_1) \\ \vdots \\ (f, b_{n-1}) \\ (f, b_n) \end{bmatrix} \qquad (4)$$

(D) Solve for the coefficients $\implies$ we have our f_a

Global spectral methods for BVPs: A least squares method

Goal: find a function, $u_a(x)$, that approximates the BVP solution $u(x)$ *accurately* on $x \in [a, b]$

(A) Pick a ***subspace*** that we want to approximate onto (e.g., $\mathscr{P}^n[a, b]$ or $\mathscr{T}^n[a, b]$)

(B) Write $u_a(x)$, as a linear combination of the basis functions in terms of unknown coefficients

$$u_a(x) = \sum_{j=0}^{n} c_j b_j(x) \qquad (5)$$

(C) Need n equations to solve for the n unknown coefficients. Seek to ***minimize the overall error***. That is, we will look for an $u_a(x)$ that satisfies

$$\min_{u_a \in \mathscr{V}} ||u - u_a||^2 \qquad (6)$$

(D) Notice that the optimal u_a satisfies

$$(u - u_a, b_i) = 0, \quad i = 0,\ldots,n \qquad (7)$$

Amazingly, all the steps to here are the same!

But now we've hit a road block, because we don't know u

Before, we knew f so this wasn't an issue

We will get around this with a specially chosen inner product

Global spectral methods: using the energy inner product

We will make progress by using the ***energy inner product*** for (7). For the 1D Poisson problem we are considering, this energy IP is defined as

$$\frac{d^2u}{dx^2} = f, \quad x \in [a, b]$$

$$(f, g)_E = \int_a^b f'(x) g'(x) dx, \quad \forall f, g \in \mathcal{V}$$

Looking at the induced norm tells us why we call this the energy IP

$$||g||_E = \sqrt{(g, g)_E} = \sqrt{\int_a^b (g'(x))^2 \, dx}$$

Remember: the 1D Poisson problem is a steady state heat equation. This is a measure of the heat flux density, $-k\frac{du}{dx}$, integrated across the domain.

Note. This energy IP is specific to the 1D Poisson equation, and has to be changed for different BVPs!

Global spectral methods: Using the energy inner product to solve for the c_j

Let's use our ***energy inner product*** for (7):

$$(u - u_a, b_i)_E = 0, \quad i = 0,\ldots,n \tag{7}$$

$$\sum_{j=0}^{n} c_j \left(b_j, b_i\right)_E = (u, b_i)_E, \quad i = 0,\ldots,n \tag{8}$$

We don't know u, but look what happens when we expand the energy IP:

$$\begin{aligned}
\left(u, b_i\right)_E &= \int_a^b u'(x) b_i'(x) dx \\
&= [u'(x) b_i(x)]_a^b - \int_a^b u''(x) b_i(x) dx \\
&= -\int_a^b u''(x) b_i(x) dx \\
&= -\int_a^b f(x) b_i(x) dx \qquad (9)
\end{aligned}$$

Integration by parts, then assume the BVP has zero Dirichlet BCs, and that the basis functions $b_i(x)$ preserve these BCs. Can handle the more general case with some tedium…

Note that this is also an IP. Let's call it $(f, b_i)_s$

We can use (9) in (8) to write out a linear system for the coefficients!

$$\sum_{j=0}^{n} c_j \left(b_j, b_i\right)_E = -(f, b_i)_s, \quad i = 0,\ldots,n$$

Does not include u!

Global spectral methods: Get the c_j $\implies$ approximate u

$$\implies \begin{bmatrix} (b_0, b_0)_E & (b_1, b_0)_E & \cdots & (b_{n-1}, b_0)_E & (b_n, b_0)_E \\ (b_0, b_1)_E & (b_1, b_1)_E & \cdots & (b_{n-1}, b_1)_E & (b_n, b_1)_E \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (b_0, b_{n-1})_E & (b_1, b_{n-1})_E & \cdots & (b_{n-1}, b_{n-1})_E & (b_n, b_{n-1})_E \\ (b_0, b_n)_E & (b_1, b_n)_E & \cdots & (b_{n-1}, b_n)_E & (b_n, b_n)_E \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = - \begin{bmatrix} (f, b_0)_s \\ (f, b_1)_s \\ \vdots \\ (f, b_{n-1})_s \\ (f, b_n)_s \end{bmatrix}$$