Lecture 15: Error in BVPs

Today:

- Continue with ***Finite difference methods*** applied to ***boundary value problems (BVPs)***
  - Characterize the error properties of our degree-2 FD method

Where are we up to now?

Last time.

(A) We developed a general protocol for developing FD methods for BVPs

(B) We wrote out the specific formula to solve for our approximate solution using a degree-2 local polynomial interpolant

$$\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f(x_2) - \frac{u_a}{\Delta x^2} \\ f(x_3) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) - \frac{u_b}{\Delta x^2} \end{bmatrix}$$

Write more succinctly as $\mathbf{A}\hat{\mathbf{u}} = \mathbf{f}$

Today. How should we expect the error of the degree-2 FD method to scale?

- Just as we saw for IVPs, we will find that the global error that we care about will again depend on notions of ***truncation error*** and ***stability.***
- We will make these notions precise in the context of BVPs today
- We will characterize the error for the specific degree 2 FD method above, but the same procedure can be extended to other methods.

Defining convergence

Our aim today is to show that an FD method will approach the true solution as we decrease Δx. That is, we want to know if the approximate solution ***converges*** to the true solution

Let's define what we mean by convergence

First, define the approximate solution and true solution at these discretization points as

$$\hat{\mathbf{u}} = \begin{bmatrix} u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \bar{\mathbf{u}} = \begin{bmatrix} u(x_2) \\ \vdots \\ u(x_n) \end{bmatrix}$$

Approximate solution at all the x_j

True solution at all the x_j

Then what we want to know is:

$$\lim_{n\to\infty} ||\bar{\boldsymbol{u}} - \hat{\boldsymbol{u}}||_2$$

That is, if we define the ***error vector*** as $\boldsymbol{e} := \bar{\boldsymbol{u}} - \hat{\boldsymbol{u}}$, then what we want to know is does the error decay to zero as $\Delta x \to 0$?

The 2 sources to error: Our old friends truncation error and stability

Notice that the error vector satisfies a special equation:

$$\begin{aligned} Ae &= A(\bar{u} - \hat{u}) \\ &= A\bar{u} - f \end{aligned}$$

(Using the equation from Slide 2)

Using this, we can say that the ***norm*** of the error is

$$\begin{aligned} ||e||_2 &= \left|\left| A^{-1}\left[A\bar{u} - f\right] \right|\right|_2 \\ &\leq \left|\left|A^{-1}\right|\right|_2 ||A\bar{u} - f||_2 \end{aligned}$$

Stability: does this inverse matrix stay bounded as $\Delta x \to 0$?

We will assume that this stays bounded by a constant for our method (see the typed notes for more details)

Truncation error: what is the error associated with trying the FD method on the true solution?

We will use Taylor series to show that our degree-2 FD method has a truncation error that scales as $O(\Delta x^2)$. (The same general procedure can be applied to other FD methods)

The punchline: by putting the results for stability and truncation error together, we have

$||\mathbf{e}||_2 \leq c\Delta x^2$ i.e., the error scales as $O(\Delta x^2)$

This scaling is specific to our method, but this approach of utilizing stability and truncation error to characterize the overall error applies to all FD methods!

Characterizing the truncation error

Analogous to the IVP setting, we define the truncation error in this BVP setting by applying our FD method to the ***true solution*** and moving all terms to the same side of the equation:

$$\boldsymbol{\tau} = \boldsymbol{A}\bar{\boldsymbol{u}} - \boldsymbol{f}.$$

Using our previous results, we can write out the j^{th} component of the truncation error as

$$\tau_j = \frac{1}{\Delta x^2}\left(u(x_{j-1}) - 2u(x_j) + u(x_{j+1})\right) - f(x_j)$$

Writing out $u(x_{j-1})$, $u(x_{j+1})$ in a Taylor series about x_j:

$$u(x_{j+1}) = u(x_j) + \Delta x u'(x_j) + \frac{\Delta x^2}{2}u''(x_j) + \frac{\Delta x^3}{6}u'''(x_j) + \frac{\Delta x^4}{24}u''''(x_j) + \cdots$$
$$u(x_{j-1}) = u(x_j) - \Delta x u'(x_j) + \frac{\Delta x^2}{2}u''(x_j) - \frac{\Delta x^3}{6}u'''(x_j) + \frac{\Delta x^4}{24}u''''(x_j) + \cdots$$

Plugging these expressions into τ_j gives

$$\tau_j = \frac{\Delta x^2}{12}u''''(x_j) + \cdots$$

Punchline: The truncation error decays at a rate of Δx^2 as $\Delta x \rightarrow 0$

Truncation error and consistency

Definition: consistency

A finite difference method that is written in the form $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$ is called *consistent* if it has a truncation error that goes to zero as $\Delta x \to 0$. If, moreover, the truncation error vanishes at a rate of Δx^r, the method is said to be consistent with order r.

Characterizing convergence in terms of stability and truncation error

Definition: convergence (take 2)

A finite difference method that is written in the form $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$ is called *convergent* if it is *stable* and *consistent*. If, moreover, the method is consistent with order r, it is called an r^{th} order method (this is because the error decays at the same rate as the truncation error by virtue of (4)).

The inequality on slide 4