Lecture 11: Multi-step Methods

Today:

- Introduce ***multi-step methods*** for solving initial value problems (still a finite difference method)

Where are we up to now?

We developed a procedure for solving IVPs of the form

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t) \qquad (1)$$

$$\mathbf{u}(t_0) = \mathbf{u}_0 \qquad (2)$$

by integrating the ODE (1):

$$\mathbf{u}(t_{k+1}) - \mathbf{u}(t_k) = \int_{t_k}^{t_{k+1}} \mathbf{f}(\mathbf{u}, t)dt$$

and using a local polynomial interpolation for $\mathbf{f}$.

e.g., FE used $\mathbf{f}(\mathbf{u}_k, t_k)$ and BE used $\mathbf{f}(\mathbf{u}_{k+1}, t_{k+1})$

Last time, we interpolated $\mathbf{f}$ using ***only*** information from $t \in [t_k, t_{k+1}]$, which led to ***one-step methods***.

This time, we will interpolate $\mathbf{f}$ using information from $t \in [t_{k-j}, t_{k+1}]$ $(j \geq 1)$, which leads to ***multi-step methods***.

Adams Bashforth methods: a class of explicit multi-step methods

One class of multi-step methods are the Adams-Bashforth methods, which interpolate $\mathbf{f}$ on $t \in [t_{k-j}, t_k]$ ($j \geq 1$)

i.e., the methods are explicit!

The general formula for an r-step Adams-Bashforth method is

$$\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \Delta t \sum_{j=k-r+1}^{k} \beta_{j-(k-r+1)} \boldsymbol{f}(\boldsymbol{u}_j, t_j)$$

This formula can feel unwieldy, so let's look at some examples:

How would we derive this? Very similar to Forward Euler, except now interpolate $\mathbf{f}$ as a line using the points t_{k-1} and t_k.

2-step (AB2)

$$\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \frac{\Delta t}{2}\left[-\boldsymbol{f}(\boldsymbol{u}_{k-1}, t_{k-1}) + 3\boldsymbol{f}(\boldsymbol{u}_k, t_k)\right]$$

3-step (AB3)

$$\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \frac{\Delta t}{12}\left[5\boldsymbol{f}(\boldsymbol{u}_{k-2}, t_{k-2}) - 16\boldsymbol{f}(\boldsymbol{u}_{k-1}, t_{k-1}) + 23\boldsymbol{f}(\boldsymbol{u}_k, t_k)\right]$$

Adams Moulton methods: a class of implicit multi-step methods

One class of multi-step methods are the Adams-Moulton methods, which interpolate $\mathbf{f}$ on $t \in [t_{k-j}, t_{k+1}]$ $(j \geq 1)$

i.e., the methods are implicit!

The general formula for an r-step Adams-Moulton method is

The same as for Adams Bashforth except now the sum goes to $k+1$ instead of k, because these methods are implicit!

$$\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \Delta t \sum_{j=k-r+1}^{k+1} \beta_{j-(k-r+1)} \boldsymbol{f}(\boldsymbol{u}_j, t_j)$$

This formula can (again) feel unwieldy, so let's look at some examples:

How would we derive this? Very similar to Forward Euler, except now interpolate $\mathbf{f}$ as a quadratic function using the points t_{k-1} , t_k, and t_{k+1}

2-step (AM2)

$$\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \frac{\Delta t}{12}\left[-\boldsymbol{f}(\boldsymbol{u}_{k-1}, t_{k-1}) + 8\boldsymbol{f}(\boldsymbol{u}_k, t_k) + 5\boldsymbol{f}(\boldsymbol{u}_{k+1}, t_{k+1}) \right]$$

3-step (AM3)

$$\begin{aligned}\boldsymbol{u}_{k+1} - \boldsymbol{u}_k = \frac{\Delta t}{24}\big[&\boldsymbol{f}(\boldsymbol{u}_{k-2}, t_{k-2}) - 5\boldsymbol{f}(\boldsymbol{u}_{k-1}, t_{k-1}) \\ &+ 19\boldsymbol{f}(\boldsymbol{u}_k, t_k) + 9\boldsymbol{f}(\boldsymbol{u}_{k+1}, t_{k+1})\big]\end{aligned}$$

A broader class of multi-step methods

A broader class of multi-step methods (that includes the Adams methods) can be expressed as

General form: multi-step methods for IVPs

An r-step method is defined by

RHS is the same as in Adams methods

$$\sum_{j=k-r+1}^{k+1} \alpha_{j-(k-r+1)} \boldsymbol{u}_j = \Delta t \sum_{j=k-r+1}^{k+1} \beta_{j-(k-r+1)} \boldsymbol{f}(\boldsymbol{u}_j, t_j) \tag{9}$$

where $\alpha_{j-(k-r+1)}, \beta_{j-(k-r+1)} \in \mathbb{R}$ and $j = k-r+1, \ldots, k+1$.

allows for a more general LHS than the $u_{k+1} - u_k$ associated with the Adams methods

Exercise. Write out the α and β coefficients for AB2

First, AB2 is a 2 step method so $r = 2$

Thus, the summation goes from $k-1$ to $k+1$ and the expression above is

$$\alpha_0 \mathbf{u}_{k-1} + \alpha_1 \mathbf{u}_k + \alpha_2 \mathbf{u}_{k+1} = \Delta t \Big(\beta_0 \mathbf{f}(\mathbf{u}_{k-1}, t_{k-1}) + \beta_1 \mathbf{f}(\mathbf{u}_k, t_k) + \beta_2 \mathbf{f}(\mathbf{u}_{k+1}, t_{k+1}) \Big)$$

Comparing this expression to that from slide 3:

$$\alpha_0 = 0,\ \alpha_1 = -1,\ \alpha_2 = 1,\ \beta_0 = -\frac{1}{2},\ \beta_1 = \frac{3}{2},\ \beta_2 = 0$$

A note: how do you "start" multi-step methods?

Consider AB2:

$$u_{k+1} - u_k = \frac{\Delta t}{2}\left[- f(u_{k-1}, t_{k-1}) + 3f(u_k, t_k)\right]$$

To get $\mathbf{u}_1$ we would need $\mathbf{u}_0$ and $\mathbf{u}_{-1}$

Doesn't exist! Time starts at t_0

So what do we do? Use a ***one step*** method of sufficient accuracy to get $\mathbf{u}_1$

Once we have $\mathbf{u}_1$, we can get $\mathbf{u}_2$ from $\mathbf{u}_1$ and $\mathbf{u}_0$. And keep advancing in time from there.

More generally, for an r-step method, use a ***one step*** method of sufficient accuracy to get $\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}$, and then let the ***multi-step*** method take over from there.